# CONVERGING SHOCK WAVE IN IDEALLY INELASTIC MEDIUM AND STABILITY OF CUMULATION

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## INTRODUCTION

The focusing of shock waves has attracted the attention of a number of scientific investigators (see the survey in [1]). The self-shaping converging state of a shock wave has been primarily the subject of their investigations. Since the focusing problem is in itself very complex, its stability has been but little studied. As far as we can ascertain there is only one article [2] on stability of a converging shock wave in which a similar approach was employed.

In the present article a boundary-value problem is considered for a converging shock wave in a cylinder or in a sphere, as well as the evolution of small multidimensional perturbations, if the motion of this wave is towards the center in an ideally inelastic medium whose material puts to the test the uniform consolidation at the front of the shock wave independent of the wave amplitude. A similar approach was employed in the analysis of the motion of a diverging shock wave [3]. The behavior of actual materials (powders, very porous bodies) is simulated in the domain of heavy loads. The solution of this kind is suitable in applications such as, for example, molding various parts from powders. From the theoretical point of view the problem is interesting as representing a case of cumulation which can be investigated quite adequately. The results are compared with those of converging acoustic shock, the comparison being significant for the following reasons. For any ideal medium a variable parameter $\varkappa$ is introduced equal to the ratio of the velocity at the front of the shock wave to the sound velocity not in the front. It should be mentioned that $0 \leq \varkappa \leq 1$ in view of a necessary condition for stability. The case of $\varkappa = 1$ corresponds to the acoustics case. The medium under our consideration is characterized by the parameter value $\varkappa = 0$. Thus the acoustic medium and the model of the ideal inelastic medium appear to be diametrically opposite cases in the asymptotics as regards the parameter $\varkappa$.

The tangential stresses can be ignored, since they are bounded by the yield limit, and the pressure amplitude of the arising strong shock wave increases as it arrives closer to the center. High temperatures arise at the front due to large losses on irreversible deformations, the yield limit vanishing in the melting state.

The solution of the symmetric convergence of a shock wave to the center or to the axis is obtained in the same way as in [3]. The asymptotics of the solution for $R \to 0$ ($R$ is the front radius) are also analyzed. It appears that the amplitude increases considerably more than in the previously studied cases of uniform media [1, 4, 5]. The system of equations for the amplitudes of the perturbation harmonics is integrated twice and reduces to a single ordinary integrodifferential equation for the perturbation of the front surface. For low consolidation the solution can be analyzed asymptotically. The equation is solved by numerical methods for consolidation parameters of the order of unity. The instability is noticed of the zeroth and the first harmonics. Other harmonics are stable.

1. Let there be a uniform sphere (or cylinder) of radius $R_0$ to whose surface one applies suddenly the pressure $P_0(t)$ at the time $t = 0$. The load instantly reaches its maximal value and then decreases monotonically. The original density of the medium is $\rho_0$. It is assumed that having reached a nonzero pressure at a point of the medium the density becomes equal to $\rho_1$ = const. This simple framework is used as a model for porous bodies in the region of high pressures. Of course, a shock wave will then spread from the

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surface towards the center. Let $r = R(t)$ and $r = R_1(t)$ be the laws of motion of the front of the shock wave and of the surface, respectively, and let U be the mass velocity.

A mathematical formulation is now given. To find the limits of the region for the motion R (t) and $R_1(t)$, as well as the functions P (r, t) and U (r, t) defined in the region $R_1(t) < r < R(t)$ which both satisfy the following equations of motion and of continuity in the interior of the region and the conditions at the front of the shock wave and on the surface:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} + \frac{1}{\rho_1}\frac{\partial P}{\partial r} = 0; \tag{1.1}$$

$$\frac{1}{r^\nu}\frac{\partial (Ur^\nu)}{\partial r} = 0; \quad R_1(t) < r < R(t);$$

$$U = \theta_0 \dot{R}; \qquad P = \rho_0\theta_0\dot{R}^2 \qquad (r = R(t));$$

$$P = P_0(t); \qquad U = \dot{R}_1 \qquad (r = R_1(t)).$$

In the above $\nu = 1, 2$ correspond to the case of a cylinder or a sphere $\theta_0 = (\rho_1 - \rho_0)/\rho_1$, $R_1(0) = R_0$, the dot at the top denoting the time derivative ($\dot{R}$ is the front velocity for the shock wave).

It follows from the continuity equation and from the first condition at the front that

$$U = \theta_0 \dot{R}(R/r)^\nu; \quad R_1^{\nu+1} = (1-\theta_0)R_0^{\nu+1} + \theta_0 R^{\nu+1}. \tag{1.2}$$

Inserting the expression for U in the first equation of (1.1) and integrating with respect to r from $r = R$ to $r = R_1$, one obtains

$$R\ddot{R} + \frac{1}{2}A_\nu \dot{R}^2 = B_\nu', \quad \left(R(0) = R_0,\ \dot{R}(0) = \sqrt{P_0(0)/\rho_0\theta_0}\right);$$

$$A_1 = 2 + \frac{2-\theta_0\left(1 + R^2/R_1^2\right)}{\ln(R/R_1)}; \quad A_2 = 4 + \frac{2-\theta_0\left(1+R^4/R_1^4\right)}{R/R_1 - 1};$$

$$B_1' = \frac{P_0(t)}{\rho_1\theta_0 \ln(R/R_1)}; \quad B_2' = \frac{P_0(t)}{\rho_1\theta_0(R/R_1 - 1)}.$$

We set $x = R/R_0$, $g = \rho_0\theta_0\dot{R}^2/P_0(0)$ and consider $P_0 = P_0(x)$ without making any change in the notation, that is, the load is given as a function of the front radius. In our case this is not an essential limitation. It is noted that any monotonically decreasing function $P_0(x)$ is associated with a monotonically decreasing function $P_0(t)$; to solve the problem for any specific load $P_0(t)$ one has to find $P_0(x)$ in such a way that this association is observed (the so-called semiinverse method). Our aim is to study the behavior of the solution for $x \to 0$ which has turned out to be independent of the boundary condition. Our assumption is not essential for a time-persisting pulse on the surface which can be simulated by a "step."

In dimensionless variables Eq. (1.2) assumes the form

$$x\frac{dg}{dx} + A_\nu g = B_\nu \quad (g(1) = 1), \tag{1.3}$$

$$B_1 = \frac{2(1-\theta_0)P_0(x)}{\ln(R/R_1)P_0(0)}; \quad B_2 = \frac{2(1-\theta_0)P_0(x)}{(R/R_1 - 1)P_0(0)}; \quad \frac{R}{R_1} = \left[\frac{1-\theta_0}{x^{\nu+1}} + \theta_0\right]^{-\frac{1}{\nu+1}}.$$

For $x \to 1$ $A_\nu$ and $B_\nu$ possess singularities: $A_1$ and $B_1$, a logarithmic singularity; $A_2$ and $B_2$, a power singularity. It can be shown that the solution of (1.3) is given by the improper integral

$$g(x) = \lim_{\varepsilon\to 0}\left\{\exp[-F_\nu(x,\varepsilon)]\int_{1-\varepsilon}^{x}\xi^{-1}B_\nu(\xi)\exp[F(\xi,\varepsilon)]\,d\xi\right\},$$

where

$$F_\nu(x,\varepsilon) = \int_{1-\varepsilon}^{x}\xi^{-1}A_\nu(\xi)\,d\xi.$$

By separating the singularities the solution can be reduced to quadratures, namely,

$$g = \frac{\exp[-G_1(x)]}{(x \ln x)^2} \int_1^x \xi (\ln \xi)^2 B_1(\xi) \exp[G_1(\xi)]\, d\xi \quad (\nu = 1); \tag{1.4}$$

$$g = \frac{\exp[-G_2(x)]}{x^2 (1-x)^2} \int_1^x \xi (1-\xi)^2 B_2(\xi) \exp[G_2(\xi)]\, d\xi \quad (\nu = 2).$$

In the above one has

$$G_1(x) = \int_1^x \left[\frac{2-\theta_0(1+\bar{\xi}^2)}{\ln \bar{\xi}} - \frac{2}{\ln \xi}\right] \frac{d\xi}{\xi};$$

$$G_2(x) = \int_1^x \left[\frac{2}{1-\xi} - \frac{2-\theta_0(1+\bar{\xi}^4)}{1-\bar{\xi}}\right] \frac{d\xi}{\xi},$$

where

$$\bar{\xi} = \xi(1-\theta_0 + \theta_0 \xi^{1+\nu})^{-1/(1+\nu)}.$$

The function U (r, x) can be found from the first relation of (1.2); to obtain P (r, x) one has to integrate the first equation of (1.1) between a point r inside the region and r = $R_1$ (and not from r = R as was done previously). For these quantities the expressions at the front follow from (1.1). They are

$$U = \sqrt{\rho_0^{-1} \theta_0 P_0(0) g(x)}, \quad P = P_0(0) g(x).$$

The function x = x (t) can be obtained from the equation

$$t = R_0 \sqrt{\frac{\rho_0 \theta_0}{P_0(0)}} \int_1^x \frac{dx'}{\sqrt{g(x')}},$$

and the remaining sought-for functions, as well as the function $P_0$, become known either as functions of the variables r, t or only of t.

The asymptotic behavior of the functions g for x → 0 follows from (1.4),

$$g \sim \frac{1}{x^2 |\ln x|^{2-\theta_0}} \quad (\nu = 1); \quad g \sim \frac{1}{x^{2+\theta_0}} \quad (\nu = 2).$$

The asymptotics of the functions P and U at the front for x → 0 are as follows:

$$P \sim \frac{1}{x^2 |\ln x|^{2-\theta_0}}; \quad U \sim \frac{1}{x |\ln x|^{1-\theta_0/2}} \quad (\nu=1); \tag{1.5}$$

$$P \sim \frac{1}{x^{2+\theta_0}}; \quad U \sim \frac{1}{x^{1+\theta_0/2}} \quad (\nu=2).$$

It should be noted here that $0 < \theta_0 < 1$.

Just as one expected, no effect is exerted on the asymptotic behavior by the boundary condition.

For the sake of comparison, the cases of a converging acoustic jump (linear acoustics) and of a converging shock wave in an ideal gas are now analyzed. The growth of pressure amplitude in the acoustic case is $P \sim x^{-1/2}$ for $\nu = 1$ and $P \sim x^{-1}$ for $\nu = 1$. In an ideal gas [4, 6] one has $P \sim x^{-k}$, where $k \approx 0.79$ for the adiabatic exponent $\gamma = 7/5$, $k \approx 0.49$ for $\gamma = 0$, $k \to 1.4$ for $\gamma \to \infty$ ($\nu = 2$). It follows from (1.5) that our case differs from the above cases in that there is a stronger singularity (cumulation degree) of the quantities at the front. It appears that such a strong cumulation is caused not by the front curvature as in the acoustics (otherwise, the index of the power would be twice as high for a sphere as for a cylinder), but by

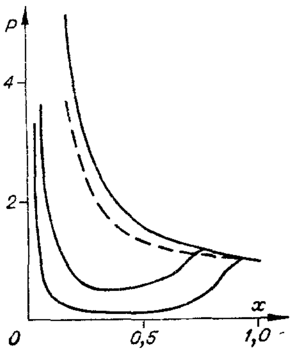

Fig. 1

the hard braking of the part of the sphere or of the cylinder actually in motion on the shock wave when the latter approaches the center.

The behavior of the functions is now analyzed inside the region for $x \to 0$. It follows from the relations for $U(r, x)$ and $P(r, x)$ that if r is kept constant and with $x \to 0$ one has

$$U \sim \frac{1}{r\,|\ln x|^{1-\theta_0/2}};\quad P \sim \frac{\ln(r/R_1)}{x^2\,|\ln x|^{3-\theta_0}}\qquad (\nu=1);$$
$$U \sim \frac{x^{1-\theta_0/2}}{r^2};\quad P \sim \frac{1-r/R_1}{x^{1+\theta_0}}\qquad (\nu=2). \tag{1.6}$$

The total kinetic energy of the medium is

$$E = 2\pi\nu\rho_1 \int\limits_{R_1}^{R} U^2 r^\nu dr$$

and for $x \to 0$ it is asymptotically

$$E \sim |\ln x|^{\theta_0-1}\quad (\nu = 1);\quad E \sim x^{1-\theta_0}\quad (\nu = 2). \tag{1.7}$$

Thus any fixed point, as well as the entire medium, is brought to a standstill with $x \to 0$ [(1.6), (1.7)].

It can be shown by energy considerations that no motion arises after focusing and that the solution is given by

$$P(r,\ t)=P_0(t);\ \ U(r,\ t)=0.$$

It follows from (1.6) that the pressure has at any point $R_1 < r < R$ a singularity for $R \to 0$. However, the interval energy density remains bounded, since the deformation remains bounded. The distribution of the specific internal energy e (per unit of mass) is given by

$$e = \frac{1}{2}\rho_0^{-1}\theta_0 P_0(0)\, g(r/R_0).$$

The above expression for the function e was obtained by taking into account that the entire interval energy in this model is thermal energy (the elastic energy is not stored) and, moreover, the heat conduction is ignored. If an additional assumption is made on the relation of the temperature T to the internal energy, then the function $T = T(r)$ can be obtained.

By way of illustration we shall carry out some calculations. In Fig. 1 graphs are shown of the pressure amplitude at the front vs the front coordinate (cylinder case). The shape of the load applied to the surface is "steplike" (the upper continuous curve) or rectangular (two neighboring continuous curves) and is of a duration which is equal to the time during which the wave passes half or a quarter of the cylinder mass, respectively. The value of $\theta_0$ was selected as 0.25. The dashed line shows the curve for $\theta_0 = 0.5$ for a steplike load. For a constant load one observes a continuous growth of pressure at the front, and the smaller $\theta$ is (that is, the more "rigid" the medium) the more rapid is the growth of amplitude due to smaller losses on irreversible deformation. However, it follows from numerical calculations, as well as from the formulas (1.5), that the cumulation is slightly stronger in a medium with a larger $\theta_0$ (this is hardly noticeable in the graph).

If the impulse is finite, then the pressure at the front grows as long as the load operates; after the load has dropped down the pressure falls very rapidly – now the damping takes place in view of irreversible losses, the minimum is reached, and, finally, it reaches the asymptotic portion in accordance with (1.5) somewhere considerably nearer the center than in the case of a "step."

The stability problem always arises [1] when unbounded cumulation is studied. In Sec. 2 a more complex problem is solved on the evolution of small disturbances downstream from the front of a shock wave moving towards the center and some conclusions are reached on the cumulation stability.

2. The effect is now considered in linear formulation of small non-one-dimensional perturbations on the motions of the medium which were studied in Sec. 1. The source of the disturbances is the boundary at which small disturbances in the applied load take place. The case of a cylindrical converging wave is considered in more detail. It is assumed that the disturbances are constant along a generator of the cylinder (variations of disturbances have, of course, no effect on the stability of cumulation). Then the system of equations for small disturbances in polar coordinates r, $\varphi$ is given by

$$\frac{\partial u'}{\partial t}+\frac{\partial (Uu')}{\partial r}+\frac{1}{\rho_1}\frac{\partial p'}{\partial r}=0;$$
$$\frac{\partial w'}{\partial t}+\frac{U}{r}\frac{\partial (w'r)}{\partial r}+\frac{1}{\rho_1 r}\frac{\partial p'}{\partial \varphi}=0;\tag{2.1}$$
$$\frac{\partial (u'r)}{\partial r}+\frac{\partial w'}{\partial \varphi}=0,\quad (R_1(t)<r<R(t)).$$

In the above p', u' and w' are small perturbations of pressure or of velocities in r and $\varphi$, respectively.

The equation of the front surface is sought for in the form

$$r=R(t)-f'(\varphi, t),$$

where $f'$ is a small perturbation of the front surface.

Similarly as in the two-dimensional case [7, 8], one can obtain from the conservation laws at the drop the conditions at the cylinder front r = R (t) of the shock wave:

$$p'=-2\rho_0\theta_0\dot{R}\frac{\partial f'}{\partial t};\quad u'=-\theta_0\frac{\partial f'}{\partial t};\tag{2.2}$$
$$w'=\frac{\theta_0\dot{R}}{R}\frac{\partial f'}{\partial \varphi}.$$

For r = $R_1$ (t) the following conditions must be satisfied:

$$p'=\dot{p}_0(\varphi, t);\quad f'=\dot{f}_0(\varphi).\tag{2.3}$$

Physically, the second condition indicates that there is a slight time divergence for the start of the external load at different points of the cylinder surface.

The problem of small perturbations beyond the converging cylindrical shock wave is then reduced to a mathematical problem of finding the functions u', w', p', $f'$ which satisfy Eqs. (2.1) and the conditions (2.2) and (2.3). The system of equations (2.1) is elliptic and a boundary-value problem is formulated for the latter.

The following change of variables is carried out:

$$u=u'R_0/r;\ w=w'r/R_0;\ p=p'R_0/(\rho_1 R\ddot{R});\ f=f'\theta_0/R_0;\tag{2.4}$$
$$h=r^2/R_0^2;\quad \tau=R^2/R_0^2;\quad \tau_1=R_1^2/R_0^2=1-\theta_0+\theta_0\tau$$

and the unknown functions are sought for in the form of Fourier expansions: u, p, $f$ in $\sin n\varphi$ and w in $\cos n\varphi$. The system of equations (2.1) and the conditions (2.2) and (2.3) for the amplitudes of the harmonics result in

$$\frac{\partial u_n}{\partial \tau}+\theta_0\frac{\partial u_n}{\partial h}+\frac{\partial p_n}{\partial h}=0;$$
$$\frac{\partial w_n}{\partial \tau}+\theta_0\frac{\partial w_n}{\partial h}+\frac{np_n}{2}=0;\quad \frac{\partial (u_n h)}{\partial h}-\frac{nw_n}{2h}=0;\tag{2.5}$$
$$(\tau_1<h<\tau);$$

$$p_n=-4(1-\theta_0)\dot{R}\frac{df_n}{d\tau};\quad u_n=-2\dot{R}\frac{df_n}{d\tau};\quad w_n=n\dot{R}f_n\quad (h=\tau);\tag{2.6}$$

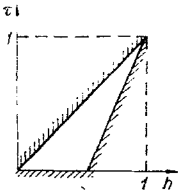


Fig. 2

$$p_n = p_{n0}(\tau); \quad f_n = f_{n0} \quad (h = \tau_1), \tag{2.7}$$

where n is the ordinal number of a harmonic (n = 0, 1, 2, ...), the variables $p_n$, $u_n$ and $w_n$ are of the dimension of the velocity, and the remaining variables are dimensionless; similarly as in the last section the load perturbation is specified as a function of the front radius, and with no change in the notation it is considered that $\dot{R} = R(\tau)$.

It is easy to change the variables (2.4), since the first two equations of (2.5) have constant coefficients and the integration domain in the plane of h, $\tau$ is a triangle (Fig. 2) $0 < \tau < 1$, $1-\theta_0 + \theta_0\tau < h < \tau$ whose sides are $h = \tau$ (the front of the shock wave), $h = 1-\theta_0 + \theta_0\tau$ (the cylinder surface), and $\tau = 0$ (the focusing instant).

One eliminates $p_n$ from the first two equations of (2.5); the obtained relations are then integrated between $u_n$ and $w_n$ from any point of the front $h = \tau$ back into the integration region along the straight line parallel to the boundary $h = 1-\theta_0 + \theta_0\tau$; finally, the function $u_n$ is eliminated from the obtained equation and the third equation of (2.5). One thus arrives at an equation for $w_n$, namely,

$$h^2 \frac{\partial^2 w_n}{\partial h^2} + h \frac{\partial w_n}{\partial h} - \frac{n^2}{4} w_n = \psi_n. \tag{2.8}$$

In the above one has

$$\psi_n = h \frac{\partial}{\partial h} \left\{ h \left[ (w_n)'_h (\eta, \eta) - \frac{n}{2} u_n (\eta, \eta) \right] \right\},$$

where

$$\eta = \frac{h - \theta_0 \tau}{1 - \theta_0}.$$

What is noticeable about Eq. (2.8) is that its left-hand side contains no derivatives with respect to $\tau$ and its right-hand side depends on the values of $w_n$ and $u_n$ at the front $h = \tau$. If one wishes to reduce the problem to the finding of a single function, one expresses $u_n$ and $dw_n/dh$ at the front in terms of $f_n$. The function $u_n(\tau, \tau)$ is related to $f_n(\tau)$ by the second relation of (2.6). Differentiating the third relation of (2.6) in the coordinate system which moves together with the undisturbed front one obtains

$$\frac{\partial w_n}{\partial \tau} + \frac{\partial w_n}{\partial h} = n \frac{d(\dot{R} f_n)}{d\tau} \qquad (h = \tau),$$

and by eliminating $dw_n/d\tau$ by virtue of the second equation of (2.5) and the first condition of (2.6) one finds

$$\frac{\partial w_n}{\partial h} = \frac{n}{1-\theta_0} \frac{d(\dot{R} f_n)}{d\tau} - 2n\ddot{R} \frac{df_n}{d\tau};$$

$$\psi_n = h \frac{\partial}{\partial h} \left\{ h \left[ \frac{n}{1-\theta_0} \frac{d(\dot{R} f_n)}{d\eta} - n\ddot{R} \frac{df_n}{d\eta} \right] \right\}.$$

One now obtains all the expressions in the square brackets as functions of $\eta$ by the formal replacement of $\tau$ by $\eta$.

The condition for $w_n$ on the cylinder surface follows from the second equation of (2.5); the former is consistent with the third condition of (2.6),

$$w_n = -\frac{n}{2} \int_1^{\tau} p_{n0}(\tau')\, d\tau' + n\dot{R}(1) f_{n0} \equiv \mu(\tau) \qquad (h = 1 - \theta_0 + \theta_0 \tau).$$

Another independent variable, $\zeta = (n \ln h)/2$, is now introduced. One has

$$A < \zeta < B: \quad \frac{\partial^2 w_n}{\partial \zeta^2} - w_n = \tilde{\psi}_n; \tag{2.9}$$

$$\zeta = A: \quad w_n = \mu_1(\tau); \quad \frac{\partial w_n}{\partial \zeta} = \mu_2(\tau); \tag{2.10}$$

$$\zeta = B: \quad w_n = \mu(\tau);$$

$$\left( f_n(1) = f_{n0}, \quad \left.\frac{df_n}{d\tau}\right|_{\tau=1} = -\frac{p_{n0}(1)}{4(1-\theta_0)\dot{R}(1)} \right).$$

In the above

$$A = \frac{n}{2}\ln\tau; \quad B = +\frac{n}{2}\ln(1-\theta_0+\theta_0\tau); \quad \mu_1 = n\dot{R}f_n;$$

$$\mu_2 = 2\tau\left[\frac{1}{1-\theta_0}\frac{d(\dot{R}f_n)}{d\tau} - 2\dot{R}\frac{df_n}{d\tau}\right]; \quad \tilde{\psi}_n[\zeta(h), \tau; f_n] = \psi_n \ (h, \tau; f_n).$$

The original problem has thus been reduced to the problem of determining two functions: $w_n(\zeta, \tau)$ and $f_n(\tau)$ which satisfy Eq. (2.9) together with the boundary conditions (2.10). The latter possess the following special features. In Eq. (2.9) there appears a derivative of $w_n$ with respect to $\tau$. The function $f_n(\tau)$ and its first derivative appear on the right of (2.9), and in the conditions (2.10) as a boundary function. The process of obtaining the solution can therefore be subdivided into two stages. Firstly, one obtains its solution by regarding (2.9) as an ordinary inhomogeneous differential equation for $w_n$. The function $f_n$ is considered as known, the variable $\tau$ is considered as a parameter, and since the number of conditions exceeds by one the number of unknowns, only the second and the third boundary conditions of (2.10) are taken into account. In the solution thus obtained $w_n$ is expressed in terms of $f_n$. Subsequently, by satisfying the first condition of (2.10) one arrives at an equation for the single function $f_n$.

The Green's function $F(\zeta, \xi)$ for Eq. (2.9) which satisfies the conditions

$$\left.\frac{\partial F}{\partial \zeta}\right|_{\zeta=A} = 0, \quad F(B, \xi) = 0;$$

is given by

$$F = \begin{cases} \dfrac{\operatorname{ch}(\zeta - A)\operatorname{sh}(\xi - B)}{\operatorname{ch}(B-A)}, & \zeta \leqslant \xi, \\ \dfrac{\operatorname{ch}(\xi - A)\operatorname{sh}(\zeta - B)}{\operatorname{ch}(B-A)}, & \zeta \geqslant \xi. \end{cases}$$

The solution of Eq. (2.9) with appropriate boundary conditions can now be written as

$$w_n = \int_A^B F(\zeta, \xi)\tilde{\psi}_n(\xi, \tau; f_n)\,d\xi + \mu_2\frac{\operatorname{sh}(\zeta - B)}{\operatorname{ch}(B-A)} + \mu\frac{\operatorname{ch}(\zeta - A)}{\operatorname{sh}(B-A)}.$$

The last of the boundary conditions (2.10) is now satisfied and after some transformations one arrives at the following integrodifferential equation for $f_n$:

$$a\frac{df_n}{d\tau} = \int_\tau^1 \left(b\frac{df_n}{d\eta} + cf_n(\eta) + d\right)d\eta; \tag{2.11}$$

$$f_n(1) = f_{n0}, \quad \left.\frac{df_n}{d\tau}\right|_{\tau=1} = -\frac{p_{n0}(1)}{4(1-\theta_0)\dot{R}(1)}.$$

In the above one has

$$a = n^{-1}\tau\dot{R}(1+\theta)\left(\alpha_*^{n/2} - \alpha_*^{-n/2}\right); \quad b = \frac{1}{2}\dot{R}\left(\alpha^{n/2} - \alpha^{-n/2}\right);$$

$$c = \frac{n(1+\theta)\dot{R}}{4(\eta+\theta\tau)}\left(\alpha^{n/2} - \alpha^{-n/2}\right); \quad d = \frac{1}{2}p_{n0}(\tau); \quad \alpha = (\eta+\theta\tau)/(1+\theta\tau);$$

$$\alpha_* = (1+\theta)\tau/(1+\theta\tau); \quad \theta = \theta_0/(1-\theta_0); \quad \dot{R} = \dot{R}(\tau).$$

Having found $f_n$ [solving (2.11) numerically is not particularly difficult] the remaining sought-for functions are now found, since they can be expressed by means of $f_n$.

The solving process is now studied for small values of the parameter $\theta$. Differentiating both sides of Eq. (2.11) with respect to $\tau$ and neglecting small quantities for $\theta \ll 1$ one obtains the equation

$$\frac{d^2\bar{f}_n}{dx^2} - \frac{1}{x}\left[1 + x\frac{d\ln(\dot{R}(x))}{dx} + \frac{2n(1+x^{2n})}{x^{2n}-1}\right]\frac{d\bar{f}_n}{dx} + \frac{n^2}{x^2}\bar{f}_n = \frac{2n\dot{R}(x)\,p_{n0}(x)}{x^2(x^{-n}-x^n)},$$

where

$$\bar{f}_n(x) = f_n(\tau(x)) = f_n(x^2).$$

The same equation could be obtained by ignoring small quantities of higher orders in Eqs. (2.1) and then reducing this simplified problem to an equation for the single function $f_n$.

Asymptotically, the function $\bar{f}_n$ for $x \to 0$ (denoted by $\tilde{f}_n$) satisfies the equation

$$x^2\frac{d^2\tilde{f}_n}{dx^2} - 2nx\frac{d\tilde{f}_n}{dx} + n^2\tilde{f}_n = 0,$$

whose general solution is given by

$$\tilde{f}_n = c_1 x^{\lambda_n^{(1)}} + c_2 x^{\lambda_n^{(2)}},$$

where $c_1$ and $c_2$ are constants and $\lambda_n^{(1),(2)} = n + \frac{1}{2} \pm \sqrt{n + \frac{1}{4}}$.

The asymptotic behavior of $f_n$ for $x \to 0$ depends on the value of the lowest exponent, that is,

$$\bar{f}_n(x \to 0) \sim x^{n+\frac{1}{2}-\sqrt{n+\frac{1}{4}}}, \qquad (2.12)$$

with $n + \frac{1}{2} - \sqrt{n + \frac{1}{4}} \geq 0$ $(n = 0, 1, 2, \ldots)$.

The stability or instability of symmetrical focusing is characterized by the ratio of the perturbation amplitude of the front surface to the front radius,

$$\frac{\bar{f}_n}{x} \sim x^{\alpha_n}; \quad \alpha_n = n - \frac{1}{2} - \sqrt{n + \frac{1}{4}} = \begin{cases} -1, & n = 0 \\ -0.618, & n = 1 \\ 0, & n = 2 \end{cases} \qquad (2.13)$$

$$\alpha_n > 0, \; n > 2.$$

The first harmonic is thus unstable, the second is relatively stable, and a harmonic $n > 2$ is absolutely stable (of course, the harmonic with $n = 0$ does in no way indicate the focusing stability). One notes at the same time that one has $\bar{f}_n \to 0$ $(n = 1, 2, \ldots, x \to 0)$, that is, the front contracts towards the center of the axis though not asymmetrically in view of $\bar{f}_1/x \to \infty$ $(x \to 0)$.

Since at the front the quantities $p_n'/P$, $u_n'/U$ and $d\bar{f}_n/dx$ are of the same order for $x \to 0$, therefore the asymptotic formulas for the relative quantities of pressure disturbance and velocity are similar to (2.13).

These results are now compared with the acoustic case. The growth of perturbations at the front of a converging acoustic jump is $\sim x^{-1}$. Consequently, in acoustics the instability manifests itself more strongly: for $n = 1$ the growth rate of disturbances is considerably higher than for the case under consideration; for $n = 1$ the singularity persists in the acoustic case but it vanishes for (2.13). In acoustics [1] the focusing is disturbed and "spreads over" a finite region. In this case the front contracts towards the center. However, due to the growth of disturbances the solution obtained loses its force and the problem of asymmetric focusing remains an open problem.

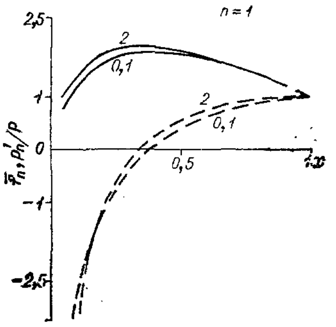


Fig. 3

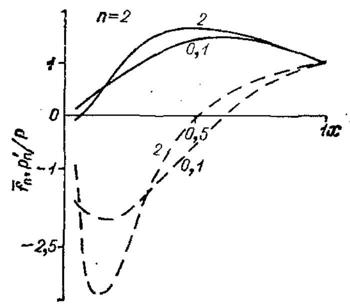


Fig. 4

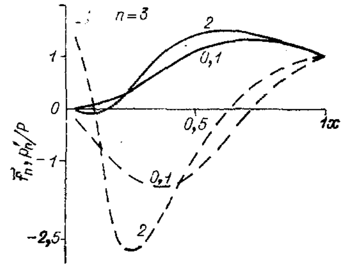


Fig. 5

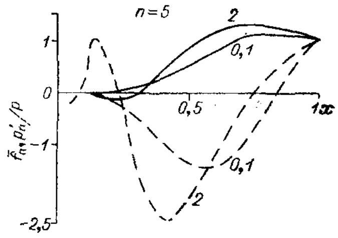


Fig. 6

To take into account the dependence of the behavior of $\bar{f}_n$ and $p_n'$ at the front on the parameter $\theta$ a program for a numerical solution of Eq. (2.11) was constructed using the finite-difference method.

The results are shown in Figs. 3-6. Continuous curves show $\bar{f}_n = \bar{f}_n(x)$, and dashed ones, $-p_n'/P = p_n'(x)/P(x)$ for the values $\theta = (\rho_1 - \rho_0)/\rho_0 = 0.1$, 2, 0. The input data for the computations were

$$\bar{f}_{n0} = 1; \quad \left.\frac{d\bar{f}_n}{dx}\right|_{x=1} = -1; \quad P_0(x) = p_{n0}(x) = \text{const.}$$

It can be seen from the graphs that for $\theta = 0.1$ the asymptotics begin for $x \approx 0.2-0.3$ being in agreement with the results of (2.12) and (2.13) valid for $\theta \ll 1$. By comparing these curves with those corresponding to the value $\theta = 2.0$ several qualitative differences can be noticed: for $\theta = 2.0$ oscillations of the perturbations can be observed. However, the stability for the values of $\theta$ of the order of unity is found similarly as in the case of $\theta \ll 1$.

To explain the perturbation effect outside the front region the disturbance was evaluated of the normal velocity of particles of the cylinder surface $u_n'$. The disturbance grows monotonically, remaining bounded for $x \to 0$. This pattern of $u_n'$ seems natural, since a fluid cylinder acted upon by an asymmetric pressure on the surface becomes large in the course of time.

The case of a sphere is briefly considered. For $\theta \ll 1$ for the asymptotics of the radial component $f_{nm}$ of the disturbance of the front surface one obtains, as in the preceding case, the equation

$$x^2 \frac{d^2\tilde{f}_{nm}}{dx^2} - (2n+1)\,x\frac{d\tilde{f}_{nm}}{dx} + n(n+1)\,f_{nm} = 0.$$

The lowest exponent $\lambda_n$ for particular solutions $x^{\lambda_n}$ of this equation is $\lambda_n = n + 1 - \sqrt{n+1}$. The asymptotic formula for $f_{nm}/x$ is given by

$$f_{nm}/x \sim x^{n - \sqrt{n+1}}.$$

For $n = 1$ this exponent is $\approx -0.43$; for $n > 1$ it is positive.

Therefore, all our conclusions as regards the focusing instability in the cylinder case remain also valid in the spherical case.

In conclusion, one should mention the following. The problem of a converging shock wave was analyzed mathematically with sufficient completeness in view of the simplicity of the chosen state equations. From the physics point of view the analyzed case is interesting, since it differs from the other analyzed cases of focusing for shock waves in homogeneous media by a high degree of cumulation. The analysis of the asymptotics of the disturbance harmonics has shown that only two harmonics can grow (the null and the first one), the singularity of the harmonic with $n = 1$ being smaller than in acoustics. Therefore, our case of strongest cumulation is at the same time of "highest stability."

It was mentioned in the introduction that the fact that the ideally inelastic and the acoustic media are limiting models of ideal media as regards the parameter $\varkappa$ enables one to put forward a hypothesis that for an ideal medium which only differs substantially from the limiting one by the value of the parameter $\varkappa$ our results are of intermediate character.

The focusing stability in a complex ideal medium was studied in an approximate manner in [2], and for low frequencies the result of unstable symmetric focusing was obtained. According to the statement made by the author in [2] this result becomes asymptotically exact for $\varkappa \to 1$. Since in our case the asymptotics in the parameter $\varkappa$ are different ($\varkappa \to 0$), the results obtained in the above-cited article and by us do not overlap.

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